The Handshake Problem

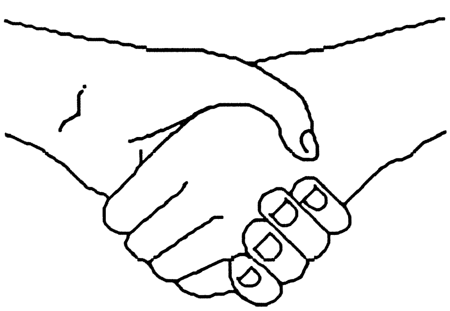
O V E R V I E W	Prerequisite Activity
This activity illustrates the use of visual thinking in mathematical problem solving. In Part I, an expression is obtained for the number of handshakes if everyone in the classroom shakes hands with one another. This expression is evaluated in Part II.	***Prerequisite Activity*** None ***Materials*** Cubes or tiles for the students (see Comment 9)

Actions

Comments

Part I

1. Mention that when people get together, they often shake hands with one another. Ask the students to each guess the number of handshakes there would be if everyone in the room shook hands with everyone else, and to record their guess on a slip of paper. Make certain the students understand what constitutes a single handshake.



1. Asking for individual guesses will encourage the students to formulate their own thoughts about the problem.

In this activity, 2 persons shaking hands is counted as 1 handshake. You can illustrate this by shaking hands with a student while saying, "This is one handshake."

2. Collect the guesses and, without comment, record them on the chalkboard or overhead.

2. This action is not essential for what follows—it may increase interest since people are often curious about others' guesses. Avoid comments on guesses that might be construed as value judgements. This will encourage students who are reluctant to make guesses for fear of being wrong, and will also discourage those who make outlandish guesses for theatrical effect.

3. Pick a student, or ask for a volunteer, to assist you. Explain to your assistant that you want her or him to help check the guesses against the actual number of handshakes.

3. The assistant will be asked to do an impossible job—this may influence who you pick.

Actions

4. Tell the students that the actual number of handshakes will now be determined. Ask them to get up and shake hands with one another. Ask your assistant to count the handshakes. Instruct your assistant to tell you if he or she has difficulty counting the handshakes.

5. Get the students' attention. Ask them to suggest procedures for shaking hands that will allow your assistant to count the number of handshakes.

Comments

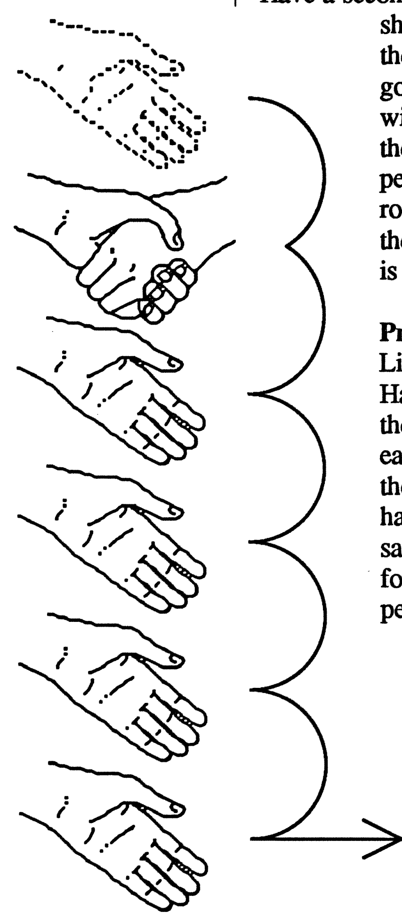
4. The students may be hesitant to start shaking hands with each other. Encourage them by moving around the room, randomly shaking hands with students. The intent is to create a setting in which it is impossible for your assistant to count all the handshakes taking place. This provides a graphic picture of the need for a systematic procedure.

If, in a minute or so, your assistant does not inform you of the hopelessness of their task, you can ask her or him if they are counting all the handshakes.

5. A number of procedures may be suggested. You may need to clarify some of the suggestions, but avoid judging one better than another.

Following are two procedures suggested by students:

Procedure A
Have a person go the center of the room. Have a second person go to the center and shake hands with the person there. Then have a third person go to the center and shake hands with the two persons already there. Continue having one person go to the center of the room and shake hands with all the people there until everyone is in the center of the room.



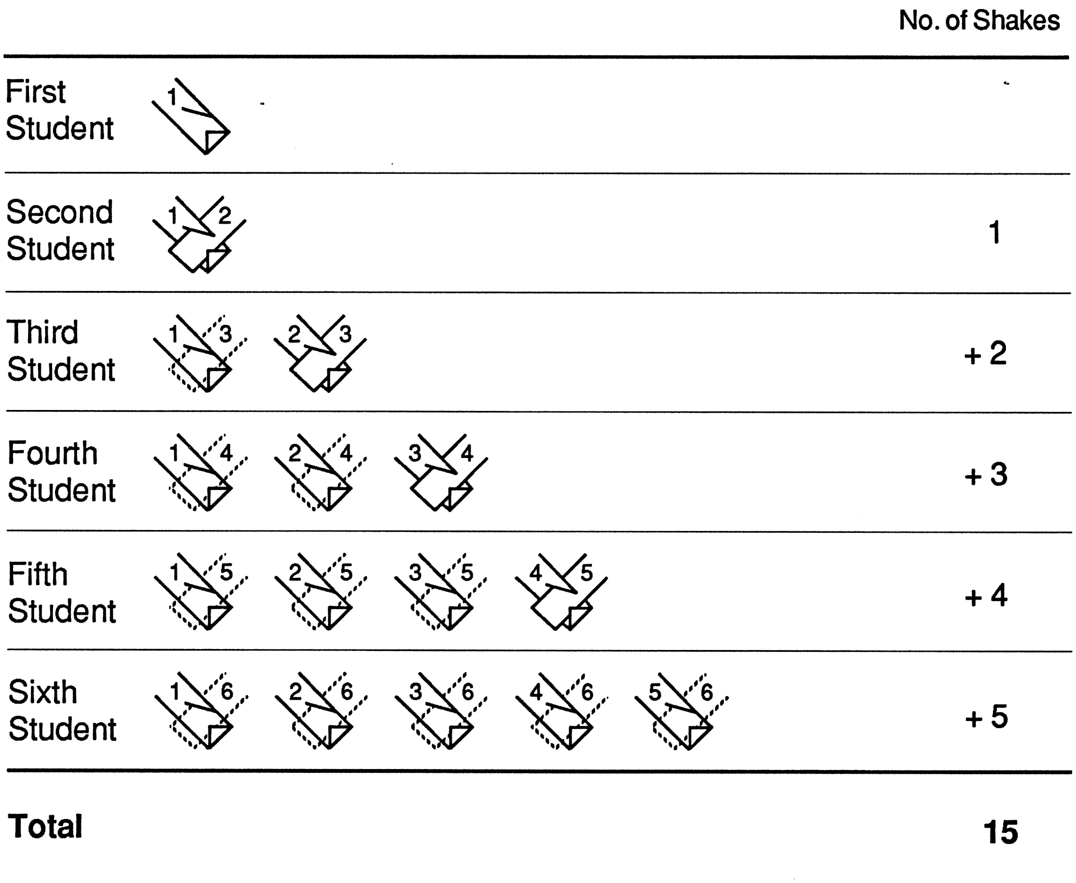
Procedure B
Line up everyone in a row. Have the first person walk down the row, shaking hands with each person, then sit down (see the illustration at left). Then have the second person do the same. Continue with the third, fourth, etc. until only one person is left in the row.

Actions

6. Use one of the procedures suggested to determine the number of handshakes if everyone in a group of 6 students shakes hands with one another. Make certain the students see that the number of handshakes for a group this size is the sum of the counting numbers from 1 through 5.

Comments

6. You may want to let the students pick a procedure to try. You can have the student who suggested the procedure carry it out with a group of students, while the rest of the class observes. The size of the group is not important, it should be large enough so students can see how the procedure would be carried out if everyone in the room participated.



Shown here is a diagram for 6 students carrying out procedure A in Comment 5. The first student goes to the center of the room. A second student goes to the center of the room and shakes hands with the student there. Then a third student goes to the center and shakes hands with the two students who are already there. This process continues until the sixth student goes to the center and shakes hands with the 5 students who are there.

As each student shakes hands, you can record the number of handshakes on the chalkboard or overhead. In this procedure, for 6 students, the number of handshakes is 1 + 2 + 3 + 4 + 5 = 15.

If procedure B in Comment 5 is carried out with a group of 6 students, the first student in the row will shake 5 hands before sitting down (see illustration on previous page). The next person will shake hands with the 4 other students remaining in the row—he or she has already shaken hands with the person sitting down. The next person will shake 3 hands, the next 2. Finally, the next to last person will shake hands once (with the last person). There is no one left for the last person to shake hands with. Thus, for 6 students, the number of handshakes is 5 + 4 + 3 + 2 +1 = 15.

Notice, in both procedures, the number of handshakes for 6 students is the sum of the first 5 positive whole numbers.

Actions

7. Discuss with the students how the number of handshakes could be determined if everyone in the room shook hands with one another.

8. (Optional.) Ask the students to answer the following questions, imagining the handshaking in their mind's eye.

When the 9 justices of the Supreme Court convene, they each shake hands with one another.

(a) How many handshakes will there be?
(b) Five of the judges shake hands with each other; then the other 4 arrive. How many more handshakes will there be?
(c) The judges form two groups. The 6 in one group have shaken hands with each other; so have the 3 in the other group. How many more handshakes will there be?

Comments

7. You can ask the students to imagine carrying out the procedure you used in Action 6 with everyone in the room participating. If the procedure used was the first one described in Comment 6 and there are 32 people in the room, the number of handshakes will be $1 + 2 + 3 + ... + 31$. (If you write an expression like this on the chalkboard or overhead, point out that '...' is a standard punctuation mark, the *ellipsis*, which, when used in mathematics, indicates something obvious has been omitted.)

Notice that for 32 people the number of handshakes is the sum of the whole numbers from 1 through 31. At this point, it is not important that students compute this sum. This will be done in Part II.

The students should see that, in general, the number of handshakes is the sum of the whole numbers beginning with 1 and ending with 1 less than the number of people shaking hands.

8. This problem can be deferred until Part II has been completed.

If students have a difficult time imagining the actions described in the problem, you can have a group of students carry them out.

(a) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$.

(b) The first of the 4 to arrive will shake 5 hands, the next 6, the next 7 and the last to arrive will shake 8 hands. So there will be $5 + 6 + 7 + 8$, or 26, more handshakes.

Students may have other ways of arriving at the answer.

(c) Each of the 6 will shake 3 hands. So there will be 6×3, or 18, more handshakes.

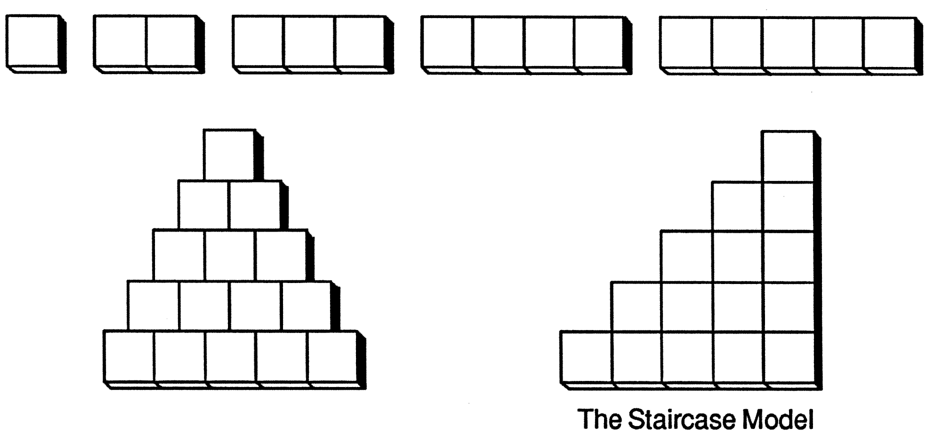
Actions

Part II

9. Distribute tile to each student. Explain to the students that they will be using tile to find sums like those encountered in the handshake problem.

10. Write '1 + 2 + 3 + 4 + 5' on the chalkboard. Ask the students to think for a few moments how they would arrange the tile to model this sum. Then ask them to make whatever arrangement came to mind. Emphasize that there is no right or wrong way to do this and you anticipate a variety of models.

11. Acknowledge, without judgement, the different models. Discuss what you see with the students. Find a "staircase" and ask the students to focus their attention on this model.



Models of 1 + 2 + 3 + 4 + 5

Comments

9. Each student will need at least 15 tile (cubes will also work).

If your supply of materials is limited, you can carry out the following actions as a demonstration using tile on the overhead or on some surface that all the students can see.

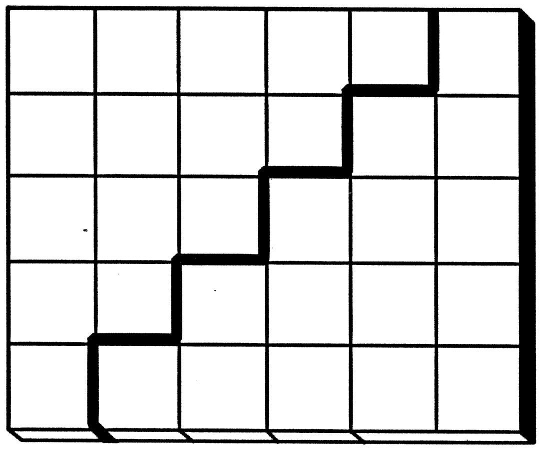
10. Asking the students to think for a few moments before arranging the tile, will help them focus on the task and not wait to see what a neighbor does.

11. Seeing the models will give you an idea of how the students relate numbers and objects. For example, some may form numerals with the tile. These students may associate school mathematics with symbols and their manipulation.

If no one forms a staircase, offer it as your model. Indicate that models are neither right nor wrong and, for particular purposes, one model may be more helpful than another. In this case, the staircase model is useful in finding the sum of consecutive whole numbers.

Actions

12. Have the students work in pairs. Ask each member of a pair to form a staircase model for the sum 1 + 2 + 3 + 4 + 5. Then ask each pair of students to form a rectangle with their two staircases. Discuss how the sum of 1 + 2 + 3 + 4 + 5 can be determined from the number of tile in this rectangle.

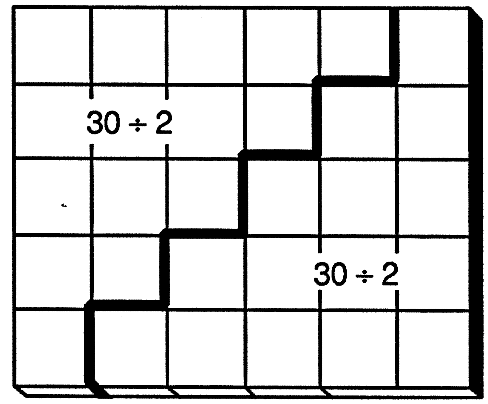


Two Staircases Form a Rectangle

13. Ask the students to imagine a staircase for the sum of the first 10 positive whole numbers, 1 + 2 + 3 + ... + 10. Ask them for the number of tile in a rectangle made from 2 of these staircases. Then ask for the number in each staircase.

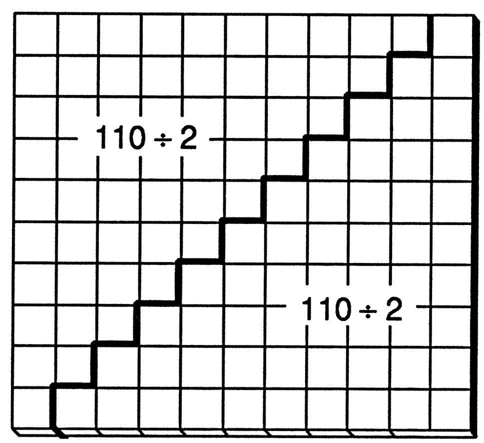
Comments

12. The rectangle contains 5 x 6, or 30, tile. Since the rectangle is comprised of 2 staircases, each staircase contains 30 ÷ 2, or 15, tile. Also, each staircase was built to contain 1 + 2 + 3 + 4 + 5 tile. Hence, 1 + 2 + 3 + 4 + 5 = 15.



1 + 2 + 3 + 4 + 5 = 30 ÷ 2 = 15

13. Two staircases form a 10 x 11 rectangul containing 110 tile. Hence each staircase contains 55 tile. Thus 1 + 2 + 3 + ... + 10 = 55.



1 + 2 + 3 + ... + 10 = 110 ÷ 2 = 55

Actions

14. Ask the students to use the 'staircase' method to find the number of handshakes if everyone in the room shakes hands with each other.

15. (Optional.)
(a) Ask the students for the number of handshakes if everyone in a room of 50 people shakes hands.
(b) Ask the students to imagine going to the classroom next door and counting the number of people in the room. Then ask them to describe how they could use this information to determine how many handshakes there would be if everyone in the classroom next door shook hands. Discuss.

Comments

14. If there are 32 people in the room, the number of handshakes will be the sum 1 + 2 + 3 + ... + 31. Two staircases, each representing this sum, will form a 31 x 32 rectangle. The number of handshakes will be half this amount: $31 \times 32 \div 2 = 496$. This computation can be done with a calculator or mentally $(31 \times 32) \div 2 = 31 \times 16 = 30 \times 16 + 1 \times 16 = 480 + 16 = 496$.

15. (a) There are $(49 \times 50) \div 2 = 1225$ handshakes. This computation is easily done with a calculator.

(b) You may want to ask the students to write instructions for computing the number of handshakes. They will have various ways of describing how to do this. Work with the students to arrive at descriptions which give correct answers and are unambiguous. Refrain from judging one correct method better than another; allow the students to make their own judgements.

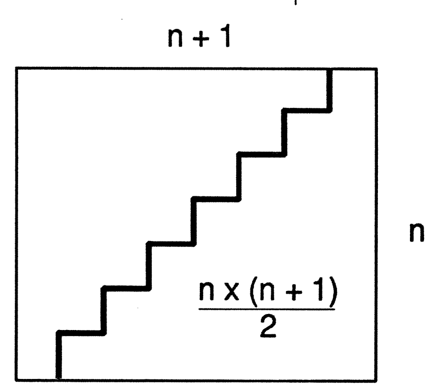
You can use this situation to show the students how formulas evolve from written descriptions. For example, [the number of handshakes] = 1 + 2 + 3 + ... + [one less than the number of people next door]. Now represent the phrases in brackets by letters: Let h stand for 'the number of handshakes' and let n stand for 'one less than the number of people next door'. Then

$$h = 1 + 2 + 3 + \dots + n.$$

This formula can be written in a simpler form: if the sum $1 + 2 + 3 + \dots + n$ is thought of as a staircase, two of them will form an $n \times (n + 1)$ rectangle. Thus

$$h = \frac{n \times (n + 1)}{2}$$

If the students are unfamiliar with the use of parentheses, explain how their use eliminates ambiguities.



$$h = 1 + 2 + 3 + \dots + n = \frac{n \times (n + 1)}{2}$$